

Backward stochastic differential equations under super linear G-expectation and associated Hamilton-Jacobi-Bellman equations *

Yuhong Xu [†]

Abstract. This paper first studies super linear G-expectation. Uniqueness and existence theorem for backward stochastic differential equations (BSDEs) under super linear expectation is established to provide probabilistic interpretation for the viscosity solution of a class of Hamilton-Jacobi-Bellman equations, including the well known Black-Scholes-Barrenblett equation, arising in the uncertainty volatility model in mathematical finance. We also show that BSDEs under super linear expectation could characterize a class of stochastic control problems. A direct connection between recursive super (sub) strategies with mutually singular probability measures and classical stochastic control problems is provided. By this result we give representation for solutions of Black-Scholes-Barrenblett equations and G-heat equations.

Key words. G-Brownian motion, super linear expectation, normal distribution, backward stochastic differential equations, Hamilton-Jacobi-Bellman equation, Feynman-Kac formula, stochastic optimization, uncertainty volatility model

AMS subject classifications. 60H10, 60H30, 60J65, 35K55, 35K05, 49L25

1 Introduction

The motive of this paper is to show that backward stochastic differential equations (BSDEs) under super linear expectation coupled with a forward diffusion:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x}) dr + \int_t^s h_j(X_r^{t,x}) d\langle B^j \rangle_r + \int_t^s \sigma_j(X_r^{t,x}) dB_r^j, \quad t \in [0, T], \quad (1.1)$$

$$Y_s^{t,x} = \mathbb{E}_*[\Phi(X_T) + \int_s^T g(X_r^{t,x}, Y_r^{t,x}) dr + \int_s^T f_j(X_r^{t,x}, Y_r^{t,x}) d\langle B^j \rangle_r | \mathcal{F}_s], \quad s \in [t, T], \quad (1.2)$$

provide a probabilistic interpretation for the viscosity solution of a class of Hamilton-Jacobi-Bellman equations (HJB):

$$\partial_t u + \inf_{\alpha \in \Gamma} \{ \mathcal{L}(x, \alpha) u + g(x, u) \} = 0 \quad (1.3)$$

$$u|_{t=T} = \Phi. \quad (1.4)$$

where $\mathcal{L}(x, \alpha)$ is a second order elliptic partial differential operator parameterized by the control variable $\alpha \in \Gamma \subset \mathbf{R}^d$,

$$\mathcal{L}(x, \alpha) = \frac{1}{2} \sum_{\mu, \nu=1}^n \left(\sum_{j=1}^d \sigma_{\mu j} \sigma_{\nu j}(x) \alpha_j^2 \right) \cdot \partial_{x^\mu x^\nu} + \sum_{i=1}^n \left(b_i(x) + \sum_{j=1}^d h_{ij}(x) \alpha_j^2 \right) \partial_{x^i} + \sum_{j=1}^d f_j(x, u) \alpha_j^2.$$

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[†]Institute of Mathematics, Shandong University, Jinan 250100, P.R.China, e-mail: xuyuhongmath@163.com.

The fact that BSDEs on the space of linear probability could provide probabilistic solutions for second order quasi-linear partial differential equation (PDE) has been studied in [13, 14, 15, 16, 27, 29, 11]. This probabilistic method was also extended to stochastic PDEs [24, 14, 25], fully nonlinear cases [17, 3, 26]. Recently, Peng [19, 20, 21, 22, 23] proposed the notion of sublinear G-expectation and established associated stochastic calculus. In fact super linear G-expectation can be introduced similarly. BSDE under super linear G-expectation (G-BSDE) is also well defined under Lipschitz condition. Generally, a new kind of BSDE corresponds to a new kind of PDE. Then a natural question comes up: for what kind of PDE, BSDE under super linear G-expectation can provide a probabilistic solution. This subject is also called Feynman-Kac formula [9, 16]. Initially, we are not sure whether G-BSDE corresponds to the HJB equation. This is the first reason that we write the present paper. Secondly, we show that G-BSDE indeed provides a new probabilistic solution of a class of HJB equations (see Peng [17] for another interpretation). This fact shows that probabilistic interpretations for solutions of HJB equation are not unique. Thirdly and more importantly, the super (sub) linear G-expectation is in fact a super (sub) strategy with mutually singular probability measures on the set of possible paths, i.e., $\mathbb{E}^*[\cdot] = \sup_{P \in \mathcal{P}} E_P[\cdot]$, $\mathbb{E}_*[\cdot] = \inf_{P \in \mathcal{P}} E_P[\cdot]$,

where \mathcal{P} is the set of risk-neutral probabilities. An already known convenient framework to deal with super strategy is the stochastic control framework. However, the connection between the super strategy problem and stochastic control is not that obvious. Recall that the stochastic control problem is the maximization of an expectation over a set of processes. By our Feynman-Kac formula, we characterize the G-expectation (G-BSDE) as a class of HJB equations, which establishes a direct and equivalent relation among super strategy (G-expectation), HJB equation and value function of a stochastic control. Superlinear G-expectation itself is important in the theory of nonlinear expectation. We develop several propositions which give deeper insight into properties of G-normal distribution and G-Brownian motion, such as Proposition 2.2~2.5. Especially in Proposition 2.4, it is proved that the quadratic variance $(\langle B \rangle_t)$ of G-Brownian motion (B_t) is differential in the sense of “quasi-surely” for each t . The dynamic programming principle is also easily obtained in our framework.

Motivated by the bid price in uncertainty volatility model [1], this paper first studies super linear G-expectation. Sublinear G-expectation and Itô calculus under which have been well studied in Peng [19, 20, 21, 22, 23]. We naturally want to know whether there is new calculus under super linear G-expectation \mathbb{E}_* . However we show that \mathbb{E}^* -Brownian motion is also a Brownian motion under \mathbb{E}_* and every super linear expectation \mathbb{E}_* is not dominated by itself but by sublinear expectation $\mathbb{E}^*[\cdot] := -\mathbb{E}_*[-\cdot]$. Due to the non-dominated property of \mathbb{E}_* , we have to work with associated sublinear G-expectation \mathbb{E}^* . In fact superlinear expectation and sublinear expectation are complementary to each other. Super linear G-expectation is an auxiliary means of sublinear G-expectation. It provides more insights to G-Brownian motion and the uncertainty of random variables under sublinear expectation. For instance, a ‘symmetric martingale’ (M_t) [28] can be characterized as that (M_t) is a martingale both under \mathbb{E}^* and \mathbb{E}_* . Recently Li and Peng [10] established a new framework for Itô integral and related stochastic calculus. However, there are many important and interesting problems still holding open under this new framework, e.g., under what condition $\int_0^\cdot \eta_s dB_s$ is a martingale or a local martingale. So we will work within Peng’s framework, which are enough for us to illustrate our subject.

We first recall some notions under nonlinear expectation and proved that a random variable X is \mathbb{E}_* -normal distributed if and only if $u(t, x) = \mathbb{E}_*[\phi(x + \sqrt{t}X)]$ is the viscosity solution of PDE $\partial_t u - \frac{1}{2} \inf_{\gamma \in \Gamma} \text{tr} \{ \gamma \gamma^T D^2 u \} = 0$, $u(0, x) = \phi(x)$. Some useful property related to super linear G-expectation are also listed. Section 3 establishes a uniqueness and existence theorem and a comparison for BSDEs under super linear G-expectation. Section 4 studies the relation between BSDEs under \mathbb{E}_* and associated HJB equation. In section 5 we show some applications: BSDEs under \mathbb{E}_* could characterize a class of stochastic control problems; we also give representation for solutions of Black-Scholes-Barrenblatt equations which arise in mathematical finance and G-heat equations which are fundamentally important in the theory of G-expectation. In the appendix we discuss the dominated convergence theorem under

sublinear expectation induced by mutually singular probability measures.

2 Super linear expectation: another point of view of sublinear expectation

For a given positive integer n we will denote by $\langle x, y \rangle$ the scalar product of $x, y \in \mathbf{R}^n$ and by $|x| = \langle x, x \rangle^{1/2}$ the Euclidean norm of x . For two stochastic processes (X_t) and (Y_t) , let $\langle X, Y \rangle_t$ denote their mutual variance. We denote by $\mathbb{S}(n)$ the collection of $n \times n$ symmetric matrices. We observe that $\mathbb{S}(n)$ is an Euclidean space with the scalar product $\langle A, B \rangle = \text{tr}[AB]$. Let Ω be a given set and let \mathcal{H} be a linear space of real functions defined on Ω such that if $X_1, \dots, X_n \in \mathcal{H}$ then $\varphi(X_1, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l.Lip}(\mathbf{R}^n)$ where $C_{l.Lip}(\mathbf{R}^n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbf{R}^n,$$

for some $C > 0$, $m \in \mathbf{N}$ depending on φ . \mathcal{H} is considered as a space of ‘random variables’. In this case $X = (X_1, \dots, X_n)$ is called an n -dimensional random vector, denoted by $X \in \mathcal{H}^n$. We also denote by $\mathcal{B}(\Omega)$ the Borel σ -algebra of Ω ; $C_b^k(\mathbf{R}^n)$ the space of bounded and k -time continuously differentiable functions with bounded derivatives of all orders less than or equal to k ; $C_{Lip}(\mathbf{R}^n)$ the space of Lipschitz continuous functions.

Definition 2.1. A **nonlinear expectation** \mathbb{E} on \mathcal{H} is a functional $\mathbb{E} : \mathcal{H} \mapsto \mathbf{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

(a) *Monotonicity:* If $X \geq Y$, then $\mathbb{E}[X] \geq \mathbb{E}[Y]$.

(b) *Constant preserving:* $\mathbb{E}[c] = c$.

If a functional $\mathbb{E}^* : \mathcal{H} \mapsto \mathbf{R}$ satisfies (a), (b) and the following

(c) *Sub-additivity:* $\mathbb{E}[X + Y] \leq \mathbb{E}[X] + \mathbb{E}[Y]$.

(d) *Positive homogeneity:* $\mathbb{E}[\lambda X] = \lambda \mathbb{E}[X], \forall \lambda \geq 0$.

then we call \mathbb{E}^* a **sublinear expectation**. If a functional $\mathbb{E}_* : \mathcal{H} \mapsto \mathbf{R}$ satisfies (a), (b), (d) and

(c') *Super-additivity:* $\mathbb{E}[X + Y] \geq \mathbb{E}[X] + \mathbb{E}[Y]$.

we call \mathbb{E}_* a **superlinear expectation**.

Definition 2.2. Let X_1 and X_2 be two n -dimensional random vectors defined on nonlinear expectation spaces $(\Omega_1, \mathcal{H}_1, \mathbb{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \mathbb{E}_2)$ respectively. They are called *identically distributed*, denoted by $X_1 \stackrel{d}{=} X_2$, if

$$\mathbb{E}_1[\varphi(X_1)] = \mathbb{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{l.Lip}(\mathbf{R}^n).$$

Definition 2.3. In a nonlinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ a random vector $Y \in \mathcal{H}^n$ is said to be independent of another random vector $X \in \mathcal{H}^m$ under \mathbb{E} if for each test function $\varphi \in C_{l.Lip}(\mathbf{R}^{m+n})$ we have

$$\mathbb{E}[\varphi(X, Y)] = \mathbb{E}[\mathbb{E}[\varphi(x, Y)]_{x=X}].$$

Remark 2.1. If Y is independent of X , then $\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$.

Definition 2.4. (*G-normal distribution with zero mean under positively homogeneous expectation*). A d -dimensional random vector $X = (X_1, \dots, X_d)$ in a positively homogeneous expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called *G-normal distributed* if for each $a, b \geq 0$ we have

$$aX + b\tilde{X} \stackrel{d}{=} \sqrt{a^2 + b^2}X,$$

where \tilde{X} is an independent copy of X .

Remark 2.2. It is easy to check that $\mathbb{E}[X] = \mathbb{E}[-X] = 0$. The so called ‘ G ’ is related to $G : \mathbb{S}(d) \mapsto \mathbf{R}$ defined by

$$G(A) = \frac{1}{2} \mathbb{E}[\langle AX, X \rangle],$$

Proposition 2.1. Let \mathbb{E}_* be a super linear expectation. $\mathbb{E}^*[\cdot] := -\mathbb{E}_*[-\cdot]$. Then

- (i) $\mathbb{E}^*[\cdot]$ is a sublinear expectation.
- (ii) There exist a family of linear expectation $\{E_P; P \in \mathcal{P}\}$ on (Ω, \mathcal{H}) such that $\mathbb{E}^*[\cdot] = \sup_{P \in \mathcal{P}} E_P[\cdot]$, $\mathbb{E}_*[\cdot] = \inf_{P \in \mathcal{P}} E_P[\cdot]$.
- (iii) $G_*(A) := \frac{1}{2} \mathbb{E}_*[\langle AX, X \rangle] = \frac{1}{2} \inf_{\gamma \in \Gamma} \text{tr} \{ \gamma \gamma^T A \}$, where Γ is a bounded and closed subset of $\mathbf{R}^{d \times d}$.
- (iv) $|\mathbb{E}_*[X|\mathcal{F}_t] - \mathbb{E}_*[Y|\mathcal{F}_t]| \leq \mathbb{E}^*[|X - Y| | \mathcal{F}_t]$.

Proof It is easy to check (i). By properties of sublinear expectation ([19, 7]), we obtain (ii) and (iii). (iv) is from Peng [18].

Definition 2.5. We introduce the natural Choquet capacity

$$C^*(A) := \sup_{P \in \mathcal{P}} P(A), \quad A \in \mathcal{B}(\Omega).$$

A property holds “quasi-surely” (q.s.) if it holds outside a polar set A , i.e., $C^*(A) = 0$. A mapping X on Ω with values in a topological space is said to be quasi-continuous (q.c.) if $\forall \varepsilon > 0$, there exists an open set O with $C^*(O) < \varepsilon$ such that $X|_{O^c}$ is continuous.

Definition 2.6. (Viscosity solution). $u \in C([0, \infty) \times \mathbf{R}^n)$ is called a viscosity subsolution (supersolution) of backward PDE

$$\partial_t u + F(t, x, u, Du, D^2 u) = 0, \quad u(T, x) = \Phi(x). \quad (2.1)$$

if for any $\varphi \in C_b^{1,3}([0, \infty) \times \mathbf{R}^n)$, any $(t, x) \in [0, \infty) \times \mathbf{R}^n$ which is a point of local maximum of $u - \varphi$,

$$\partial_t \varphi + F(t, x, \varphi, D\varphi, D^2 \varphi) \geq 0, \quad (\text{resp. } \leq 0). \quad (2.2)$$

u is called a viscosity solution of PDE (2.1) if it is both super and subsolution.

Remark 2.3. If PDE (2.1) is in forward form [4], the signs ‘+’, ‘ \geq ’, ‘ \leq ’ in (2.2) are changed into ‘−’, ‘ \leq ’, ‘ \geq ’ respectively. Compared with definition of viscosity solution in [4, 8, 17], we replace $C^{1,2}$ by $C_b^{1,3}$ just for technical convenience. In fact there are trivial difference between these two definitions.

Proposition 2.2. A random variable X is G_* -normal distributed if and only if

$$u(t, x) = \mathbb{E}_* \left[\varphi(x + \sqrt{t}X) \right], \quad (t, x) \in [0, \infty) \times \mathbf{R}^d, \varphi \in C_{l.Lip}(\mathbf{R}^n), \quad (2.3)$$

is the viscosity solution of PDE

$$\partial_t u - G_*(D^2 u) = 0, \quad u(0, x) = \varphi(x). \quad (2.4)$$

where $D^2 u = (\partial x^i \partial x^j u)_{i,j=1}^d$, $G_*(A) = \frac{1}{2} \mathbb{E}_*[\langle AX, X \rangle] = \frac{1}{2} \inf_{\gamma \in \Gamma} \text{tr} \{ \gamma \gamma^T A \}$, Γ is a bounded and closed subset of $\mathbf{R}^{d \times d}$.

Proof We replace t by $T-t$ in (2.3) and PDE(2.4), then the necessity is an immediate result of Theorem 4.1 in the present paper. It can be also proved similarly as in Peng [20, 23].

Conversely, let Y be a G_* -normal distributed random variable. Then $\forall \varphi \in C_{l.Lip}(\mathbf{R}^n)$, $v(t, x) = \mathbb{E}_* [\varphi(x + \sqrt{t}Y)]$ is a viscosity solution of PDE (2.4). If $\forall \varphi \in C_{l.Lip}(\mathbf{R}^n)$, $u(t, x) = \mathbb{E}_* [\varphi(x + \sqrt{t}X)]$ is also a viscosity solution of PDE (2.4), then by the uniqueness of viscosity solution of PDE (2.4) (see [4, 8]), we get that $v(t, x) = u(t, x)$, $\forall (t, x) \in [0, \infty) \times \mathbf{R}^d$. Therefore we obtain that $X \stackrel{d}{=} Y$, thus X is G_* -normal distributed. \square

Remark 2.4. The above proposition also holds under the framework of sublinear expectation.

Definition 2.7. (G -Brownian motion). A d -dimensional process $(B_t)_{t \geq 0}$ on a nonlinear expectation space $(\Omega, \mathcal{H}, \mathbb{E})$ is called a G -Brownian motion if the following properties are satisfied:

- (i) $B_0(\omega) = 0$;
- (ii) For each $t, s \geq 0$, the increment $B_{t+s} - B_t$ is independent from $(B_{t_1}, B_{t_2}, \dots, B_{t_n})$, for each $n \in \mathbf{N}$ and $0 \leq t_1 \leq \dots \leq t_n \leq t$;
- (iii) $B_{t+s} - B_t \stackrel{d}{=} \sqrt{s}X$, where X is G -normal distributed.

Proposition 2.3. Let (B_t) be a one-dimensional G^* -Brownian motion. $(\langle B \rangle_t)$ denotes its quadratic variance. Then

(i) $(\langle B \rangle_t)$ is a continuous, increasing process with finite variance, independent and stationary increments under \mathbb{E}^* .

(ii)

$$\mathbb{E}^*[\varphi(\langle B \rangle_{t+s} - \langle B \rangle_s) | \mathcal{F}_s] = \sup_{\underline{\sigma}^2 \leq \alpha^2 \leq \bar{\sigma}^2} \varphi(\alpha^2 t), \forall \varphi \in C(\mathbf{R}). \quad (2.5)$$

$$\mathbb{E}_*[\varphi(\langle B \rangle_{t+s} - \langle B \rangle_s) | \mathcal{F}_s] = \inf_{\underline{\sigma}^2 \leq \alpha^2 \leq \bar{\sigma}^2} \varphi(\alpha^2 t), \forall \varphi \in C(\mathbf{R}). \quad (2.6)$$

where we denote the usual parameters $\bar{\sigma}^2 = \mathbb{E}^*[\langle B \rangle_{t=1}]$, $\underline{\sigma}^2 = \mathbb{E}_*[\langle B \rangle_{t=1}]$.

(iii) For each $0 \leq t \leq T < \infty$, we have

$$\underline{\sigma}^2(T-t) \leq \langle B \rangle_T - \langle B \rangle_t \leq \bar{\sigma}^2(T-t) \text{ q.s..} \quad (2.7)$$

Therefore for (η_s) such that $\int_0^t |\eta_s|^2 ds < \infty$, q.s., we have

$$\underline{\sigma}^2 \int_0^t |\eta_s|^2 ds \leq \int_0^t |\eta_s|^2 d\langle B \rangle_s \leq \bar{\sigma}^2 \int_0^t |\eta_s|^2 ds, \text{ q.s..} \quad (2.8)$$

Proof See Peng [19, 20, 22, 23] for (i). $\forall \varphi \in C_{l.Lip}(\mathbf{R})$, (2.5) and (2.6) hold true. Note that any $\varphi \in C(\mathbf{R})$ can be approximated by $\varphi_n \in C_{l.Lip}(\mathbf{R})$ uniformly on a bounded subset of \mathbf{R} . Thus $\forall \varphi \in C(\mathbf{R})$, we have $\mathbb{E}_*[\varphi(\langle B \rangle_{t+s} - \langle B \rangle_s) | \mathcal{F}_s] = \lim_{n \rightarrow \infty} \mathbb{E}_*[\varphi_n(\langle B \rangle_{t+s} - \langle B \rangle_s) | \mathcal{F}_s] = \lim_{n \rightarrow \infty} \min_{\underline{\sigma}^2 \leq \alpha^2 \leq \bar{\sigma}^2} \varphi_n(\alpha^2 t) = \min_{\underline{\sigma}^2 \leq \alpha^2 \leq \bar{\sigma}^2} \varphi(\alpha^2 t)$. (2.7) is from [23]. (2.8) is a sequence of (2.7). \square

Proposition 2.4. Let \mathbb{E}_* be a super linear expectation. $\mathbb{E}^*[\cdot] := -\mathbb{E}_*[-\cdot]$. Let (B_t) be a one-dimensional G^* -Brownian motion under \mathbb{E}^* . Then

(i) (B_t) and $(-B_t)$ are Brownian motion under \mathbb{E}_* .

(ii) For each linear expectation $E_P, P \in \mathcal{P}$ on (Ω, \mathcal{H}) , (B_t) is a E_P -martingale and there exist an \mathcal{F}_t^W -adapted process (z_t) such that $E_P \int_0^t |z_s|^2 ds < \infty$ and $B_t = \int_0^t z_s dW_s$, (W_t) is a standard E_P -Brownian motion with $\underline{\sigma}^2 \leq z_t^2 \leq \bar{\sigma}^2$, for a.e. t, P -a.s..

(iii) $\frac{d\langle B \rangle_t}{dt}$ exists q.s. for each $t \geq 0$ and $\underline{\sigma}^2 \leq \frac{d\langle B \rangle_t}{dt} \leq \bar{\sigma}^2$.

(iv) $\mathbb{E}^*[\varphi(\frac{d\langle B \rangle_t}{dt})] = \sup_P \varphi(v_P)$, $\forall \varphi \in C(\mathbf{R})$.

Proof (i) We only check (iii) in Definition 2.7. Since $(B_{t+s} - B_t) \stackrel{d}{=} \sqrt{s}X$, X is a G^* -normal distributed random variable. So $\mathbb{E}_*[\varphi(B_{t+s} - B_t)] = -\mathbb{E}_*[-\varphi(B_{t+s} - B_t)] = -\mathbb{E}_*[-\varphi(\sqrt{s}X)] = \mathbb{E}_*[\varphi(\sqrt{s}X)]$, $\forall \varphi \in C_{l.Lip}(\mathbf{R})$. On the other hand, it is easy to check that X is also \mathbb{E}_* -normal distributed. By Definition 2.7, (B_t) is a \mathbb{E}_* -Brownian motion. Similarly $(-B_t)$ is also a Brownian motion under \mathbb{E}_* .

(ii) It is easy seen that $\mathbb{E}^*[\cdot|\mathcal{F}_t] = \sup_{P \in \mathcal{P}} E_P[\cdot|\mathcal{F}_t]$ and $\mathbb{E}_*[\cdot|\mathcal{F}_t] = \inf_P E_P[\cdot|\mathcal{F}_t]$. For each $P \in \mathcal{P}$, $E_P[B_{t+s} - B_t|\mathcal{F}_t] \leq \mathbb{E}^*[B_{t+s} - B_t|\mathcal{F}_t] = 0$ and $E_P[B_{t+s} - B_t|\mathcal{F}_t] \geq \mathbb{E}_*[B_{t+s} - B_t|\mathcal{F}_t] = 0$. So we have $E_P[B_{t+s}|\mathcal{F}_t] = B_t$. By the classical martingale representation, there exist an \mathcal{F}_t^W -adapted process (z_t) such that $E_P \int_0^t |z_s|^2 ds < \infty$ and $B_t = \int_0^t z_s dW_s$, (W_t) is a standard E_P -Brownian motion. By Proposition 2.3(iii), we have $\underline{\sigma}^2 t \leq \langle B \rangle_t = \int_0^t |z_s|^2 ds \leq \bar{\sigma}^2 t$ for any t , thus $\underline{\sigma}^2 \leq |z_t|^2 \leq \bar{\sigma}^2$.

(iii) By (ii) we know that for each $P \in \mathcal{P}$, $\langle B \rangle_t$ is P -a.s. differential at any t . For each given t , let A denotes the set of $\omega \in \Omega$ such that $\lim_{s \rightarrow t} \frac{\langle B^j \rangle_s - \langle B^j \rangle_t}{s-t}$ does not exist. Then we obtain that for each $P \in \mathcal{P}$, $P(A) = 0$ and $C^*(A) = \sup_{P \in \mathcal{P}} P(A) = 0$. Therefore $\frac{d\langle B \rangle_t}{dt}$ exists $q.s.$ for each $t \geq 0$.

(iv) By (ii), for each $P \in \mathcal{P}$, there is a constant $v_P \in [\underline{\sigma}^2, \bar{\sigma}^2]$ such that $E_P[\varphi(\frac{d\langle B \rangle_t}{dt})] = E_P[\varphi(z_t^2)] = \varphi(v_P)$, so $\mathbb{E}^*[\varphi(\frac{d\langle B \rangle_t}{dt})] = \sup_{P \in \mathcal{P}} E_P[\varphi(\frac{d\langle B \rangle_t}{dt})] = \sup_P \varphi(v_P)$, $\forall \varphi \in C(\mathbf{R})$. \square

Every G-Brownian motion is related to a sub (super) linear function G and a nonempty, bounded and closed subset Γ of $\mathbf{R}^{d \times d}$. Γ characterizes the variance uncertainty of corresponding G-Brownian motion. One should note that for a general Γ , components of G_Γ -Brownian motion may be not independent from others. So the item $\langle B^i, B^j \rangle_t$ arises in the G-Itô formula and G-stochastic differential equation.

Consider the following typical nonlinear heat equation:

$$\partial_t u(t, x) - \frac{1}{2} \sum_{i=1}^d [\underline{\sigma}_i^2 (\partial_{x^i x^i} u(t, x))^+ - \bar{\sigma}_i^2 (\partial_{x^i x^i} u(t, x))^-] = 0. \quad (2.9)$$

The above equation corresponds to $\Gamma = \{diag[\gamma_1, \dots, \gamma_d], \gamma_i^2 \in [\underline{\sigma}_i^2, \bar{\sigma}_i^2], i = 1, \dots, d\}$. Let $(B_t) = (B_t^1, \dots, B_t^d)$ be a d -dimensional G-Brownian motion, then

Proposition 2.5. *If (B_t^j) is independent of (B_t^i) , $1 \leq i < j \leq d$, then*

- (i) $u(t, x) = \mathbb{E}_*[\phi(x + B_t)]$ is the viscosity solution of PDE (2.8) with $u(0, x) = \phi(x)$.
- (ii) $\Gamma = \{diag[\gamma_1, \dots, \gamma_d], \gamma_i^2 \in [\underline{\sigma}_i^2, \bar{\sigma}_i^2], i = 1, \dots, d\}$.

Proof For convenience, we only consider two dimensional G-Brownian motion. If (B_t^2) is independent of (B_t^1) , then similarly to Proposition 2.2, we can prove that $u(t, x_1, x_2) = \mathbb{E}_*[\phi(x_1 + B_t^1, x_2 + B_t^2)]$ is the viscosity solution of PDE (2.8), which leads to that Γ is a set of diagonal matrices. \square

Let $\Omega = C_0^d(\mathbf{R}^+)$ denote the space of all \mathbf{R}^d -valued continuous paths $(\omega_t)_{t \in \mathbf{R}^+}$ with $\omega_0 = 0$, by $C_b(\Omega)$ all bounded and continuous functions on Ω . For each fixed $T \geq 0$, we consider the following space of random variables:

$$C_{l.Lip}(\Omega_T) := \{X(\omega) = \varphi(\omega_{t_1 \wedge T}, \dots, \omega_{t_m \wedge T}), \forall m \geq 1, \forall \varphi \in C_{l.Lip}(\mathbf{R}^m)\}.$$

We also denote

$$C_{l.Lip}(\Omega) := \bigcup_{n=1}^{\infty} C_{l.Lip}(\Omega_n).$$

We will consider the canonical space and set $B_t(\omega) = \omega_t$. For a given sublinear function $G^*(A) = \frac{1}{2} \sup_{\gamma \in \Gamma} \text{tr} \{ \gamma \gamma^T A \}$, where $A \in \mathbb{S}(d)$, Γ is a given nonempty, bounded and closed subset of $\mathbf{R}^{d \times d}$, by the following

$$\partial_t u(t, x) - G^*(D_x^2 u) = 0, \quad u(0, x) = \varphi(x),$$

Peng [19] defined G^* -expectation \mathbb{E}^* as $\mathbb{E}^*[\varphi(x + B_t)] = u(t, x)$. For each $p \geq 1$, $X \in C_{l.Lip}(\Omega)$, $\|X\|_p = \mathbb{E}^*[|X|^p]^{\frac{1}{p}}$ forms a norm and \mathbb{E}^* can be continuously extended to a Banach space, denoted by $L_G^p(\Omega)$. Hu and Peng [7] proved that $L_G^p(\Omega) = \{X \mid X \text{ is } C^* \text{-quasi-continuous, } \mathcal{B}(\Omega) \text{-measurable function s.t. } \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty\}$. By the method of Markov chains, Peng [19, 21] also defined corresponding conditional expectation, $\mathbb{E}^*[\cdot | \mathcal{F}_t] : L_G^1(\Omega) \mapsto L_G^1(\Omega_t)$, where $\mathcal{F}_t := \mathcal{B}(\Omega_t)$, $\Omega_t := \{\omega \cdot \wedge_t : \omega \in \Omega\}$. Under $\mathbb{E}^*[\cdot]$, the canonical process $B_t(\omega) = \omega_t$, $t \in [0, \infty)$ is a G^* -Brownian motion. The G^* -expectation \mathbb{E}^* can be extended to more general space. We define the upper expectation for each $\mathcal{B}(\Omega)$ -measure random variable which makes the following definition meaningful:

$$\bar{\mathbb{E}}[X] = \sup_{P \in \mathcal{P}} E_P[X].$$

Note that $\bar{\mathbb{E}} = \mathbb{E}^*$ on $L_G^p(\Omega)$ and $\forall A \in \mathcal{B}(\Omega)$, $\bar{\mathbb{E}}[\mathbf{1}_A] = C^*(A)$. Without the loss of generality, we still denote $\bar{\mathbb{E}}$ by \mathbb{E}^* .

Similarly, we can define superlinear expectation, $\mathbb{E}_*[\cdot]$ and $\mathbb{E}_*[\cdot | \mathcal{F}_t]$ by the following PDE

$$\partial_t u(t, x) - G_*(D_x^2 u) = 0, \quad u(0, x) = \varphi(x). \quad (2.10)$$

where $G_*(A) = \frac{1}{2} \inf_{\gamma \in \Gamma} \text{tr} \{ \gamma \gamma^T A \}$. It is easy to check that $\mathbb{E}_*[\cdot] = -\mathbb{E}^*[-\cdot]$ and $\mathbb{E}_*[\cdot | \mathcal{F}_t] = -\mathbb{E}^*[-\cdot | \mathcal{F}_t]$. The following property hold for $\mathbb{E}_*[\cdot | \mathcal{F}_t]$ q.s..

Proposition 2.6. *For $X, Y \in L_G^1(\Omega)$, we have q.s.,*

- (i) $\mathbb{E}_*[\eta X | \mathcal{F}_t] = \eta^+ \mathbb{E}_*[X | \mathcal{F}_t] + \eta^- \mathbb{E}_*[-X | \mathcal{F}_t]$, for bounded $\eta \in L_G^1(\Omega_t)$.
- (ii) If $\mathbb{E}_*[X | \mathcal{F}_t] = -\mathbb{E}_*[-X | \mathcal{F}_t]$, for some t , then $\mathbb{E}_*[X + Y | \mathcal{F}_t] = \mathbb{E}_*[X | \mathcal{F}_t] + \mathbb{E}_*[Y | \mathcal{F}_t]$.
- (iii) $\mathbb{E}_*[X + \eta | \mathcal{F}_t] = \mathbb{E}_*[X | \mathcal{F}_t] + \eta$, $\eta \in L_G^1(\Omega_t)$.

For a partition of $[0, T]$: $0 = t_0 < t_1 < \dots < t_N = T$, we set

$\mathcal{M}_G^{p,0}(0, T)$: the collection of processes $\eta_t(\omega) = \sum_{j=0}^N \xi_j(\omega) \cdot \mathbf{1}_{[t_j, t_{j+1}]}(t)$, where $\xi_j(\omega) \in L_G^p(\Omega_{t_j})$, $j = 0, 1, \dots, N$.

$\mathcal{M}_G^p(0, T)$: the completion of $\mathcal{M}_G^{p,0}(0, T)$ under norm $\|\eta\| = \left(\mathbb{E}^* \left[\int_0^T |\eta_t|^p dt \right] \right)^{\frac{1}{p}}$.

$\mathcal{H}_G^p(0, T)$: the completion of $\mathcal{M}_G^{p,0}(0, T)$ under norm $\|\eta\|_* = \left(\int_0^T \mathbb{E}^* [|\eta_t|^p] dt \right)^{\frac{1}{p}}$.

Itô integral for process $\eta_t \in \mathcal{M}_G^2(0, T)$ is well defined in Peng [23]. A new framework of Itô integral is constructed in Li and Peng [10]. An important difference between these two integral is that integrands in the latter Itô integral may be not quasi-continuous. Itô formula have been obtained in Peng [19, 21, 23], Gao [2] and generalized by Li and Peng [10]. We now adapt Li and Peng's formula to our framework. In fact Gao's and Peng's are enough for us to use.

Proposition 2.7. *For an Itô process $X_t = x_0 + \int_0^t b_s ds + \int_0^t \eta_s^{ij} d\langle B^i, B^j \rangle_s + \int_0^t \beta_s^j dB_s^j$, where $x_0 \in \mathbf{R}^n$, $b_s, \eta_s^{i,j}, \beta_s^j \in \mathcal{M}_G^2(0, T; \mathbf{R}^n)$ and $\Phi \in C^{1,2}([0, \infty) \times \mathbf{R}^n \mapsto \mathbf{R})$ s.t. $\partial \Phi_t$, $\partial_{x^\nu} \Phi b_s^\nu$, $\partial_{x^\nu} \Phi \eta_s^{\nu ij}$, $\partial_{x^\mu x^\nu} \Phi \beta_s^{\nu i} \beta_s^{\nu j} \in \mathcal{M}_G^1(0, T; \mathbf{R})$, $\partial_{x^\nu} \Phi \beta_s^{\nu j} \in \mathcal{M}_G^2(0, T; \mathbf{R})$, then for each t , we have q.s.,*

$$\begin{aligned} \Phi(t, X_t) &= \Phi(0, x_0) + \int_0^t \partial_{x^\nu} \Phi \beta_s^{\nu j} dB_s^j + \int_0^t (\partial \Phi_t + \partial_{x^\nu} \Phi b_s^\nu) ds \\ &\quad + \int_0^t (\partial_{x^\nu} \Phi \eta_s^{\nu ij} + \frac{1}{2} \partial_{x^\mu x^\nu} \Phi \beta_s^{\nu i} \beta_s^{\nu j}) d\langle B^i, B^j \rangle_s. \end{aligned}$$

Here we use the Einstein convention, i.e., the above repeated indices μ, ν, i, j imply summation.

3 BSDEs under super linear expectation and comparison

We consider the following n -dimensional backward stochastic differential equation (BSDE)

$$Y_t = \mathbb{E}_* \left[\xi + \int_t^T g(s, Y_s) ds + \int_t^T f(s, Y_s) d\langle B \rangle_s | \mathcal{F}_t \right] \quad (3.1)$$

where

(H3.1). $\xi \in L_G^1(\mathcal{F}_T)$.

(H3.2). There exists a constant $K \geq 0$, such that for a.e.t, q.s., $\forall y^1, y^2, z^1, z^2$:

$$|g(t, y^1) - g(t, y^2)| \leq K|y^1 - y^2|, \quad |f(t, y^1) - f(t, y^2)| \leq K|y^1 - y^2|,$$

and the process $(g(t, 0))_{t \in [0, T]}$ and $(f(t, 0))_{t \in [0, T]} \in \mathcal{M}_G^1(0, T)$.

Theorem 3.1. *There is a unique solution $(Y_t) \in \mathcal{H}_G^1(0, T; \mathbf{R}^n)$ for (3.1).*

Proof Consider a mapping Λ from \mathcal{M}_G^1 to \mathcal{H}_G^1 defined as follows, for $Y \in \mathcal{M}_G^1$,

$$\Lambda_t(Y) = \mathbb{E}_* \left[\xi + \int_t^T g(s, Y_s) ds + \int_t^T f(s, Y_s) d\langle B \rangle_s | \mathcal{F}_t \right].$$

It is easy to deduce from Lipschitz conditions that indeed $\Lambda_t(Y) \in \mathcal{H}_G^1$. A solution of (3.1) is a fixed point of the mapping Λ . Uniqueness and existence of a fixed point will follow the fact that, for each $T > 0$, Λ is a strict contraction on $\mathcal{H}_G^{1, \beta}$ equipped with the norm $\|X\|_\beta = \left[\int_0^T e^{-2\beta t} \mathbb{E}^* |X_t|^2 dt \right]^{1/2}$ for β large enough.

Let $Y, Y' \in \mathcal{H}_G^1$, by Proposition 2.1(iv), we have

$$\begin{aligned} \mathbb{E}^* |\Lambda_t(Y) - \Lambda_t(Y')| &\leq \mathbb{E}^* \left| \mathbb{E}_* \left[\xi + \int_t^T g(s, Y_s) ds + \int_t^T f(s, Y_s) d\langle B \rangle_s | \mathcal{F}_t \right] \right. \\ &\quad \left. - \mathbb{E}_* \left[\xi + \int_t^T g(s, Y'_s) ds + \int_t^T f(s, Y'_s) d\langle B \rangle_s | \mathcal{F}_t \right] \right| \\ &\leq \mathbb{E}^* \left| \int_t^T g(s, Y_s) - g(s, Y'_s) ds + \int_t^T f(s, Y_s) - f(s, Y'_s) d\langle B \rangle_s \right| \\ &\leq \mathbb{E}^* \int_t^T |g(s, Y_s) - g(s, Y'_s)| ds + \bar{\sigma}^2 \mathbb{E}^* \int_t^T |f(s, Y_s) - f(s, Y'_s)| ds \\ &\leq (1 + \bar{\sigma}^2) K \mathbb{E}^* \int_t^T |Y_s - Y'_s| ds \end{aligned}$$

Let $\beta = K(1 + \bar{\sigma}^2)$. Multiply $e^{2\beta t}$ on both sides of above inequality and then integrate them on $[0, T]$, we deduce that $\|\Lambda_t(Y) - \Lambda_t(Y')\|_\beta \leq \frac{1}{2} \|Y - Y'\|_\beta$. Note that \mathcal{H}_G^1 is a sub space of \mathcal{M}_G^1 , thus by the contract mapping principle and the equivalent of norms, there is a unique fixed point for operator Λ under norm $\|\cdot\|_* = \int_0^T \mathbb{E}^* |\cdot|^2 dt$. \square

For the following one dimensional linear BSDE:

$$Y_s = \mathbb{E}_* \left[\xi + \int_t^T (a_s Y_s + A_s) ds + \int_t^T (b_s Y_s + C_s) d\langle B \rangle_s | \mathcal{F}_t \right], \quad t \in [0, T], \quad (3.2)$$

Assuming **(H3.1)** and

(H3.2)'. There exists a constant $K \geq 0$, such that for a.e.t, q.s.,

$$|a_t| + |b_t| \leq K,$$

and the process $(A_t)_{t \in [0, T]}$ and $(C_t)_{t \in [0, T]} \in \mathcal{M}_G^1(0, T; \mathbf{R})$.

Theorem 3.2. *The following process*

$$Y_t = Q_t^{-1} \mathbb{E}_* \left[Q_T \xi + \int_t^T Q_s A_s ds + \int_t^T Q_s C_s d\langle B \rangle_s | \mathcal{F}_t \right] \quad (3.3)$$

solves BSDE (3.2), where

$$Q_t = \exp \left\{ \int_0^t a_s ds + \int_0^t b_s d\langle B \rangle_s \right\}.$$

Proof By G-Itô formula, we have

$$dQ_t = d \exp \left\{ \int_0^t a_s ds + \int_0^t b_s d\langle B \rangle_s \right\} = Q_t a_t dt + Q_t b_t d\langle B \rangle_t.$$

We denote $M_t := \mathbb{E}_* \left[\xi + \int_0^T (a_s Y_s + A_s) ds + \int_0^T (b_s Y_s + C_s) d\langle B \rangle_s | \mathcal{F}_t \right]$. Then by 2-dimensional G-Itô formula (It holds similarly to the classical Itô formula w.r.t semi-martingale) we get that

$$\begin{aligned} d(Q_t Y_t) &= Q_t dM_t - Q_t [(a_t Y_t + A_t) dt + (b_t Y_t + C_t) d\langle B \rangle_t] + Q_t a_t Y_t dt + Q_t b_t Y_t d\langle B \rangle_t \\ &= Q_t dM_t - Q_t A_t dt - Q_t C_t d\langle B \rangle_t \end{aligned}$$

Therefore

$$Q_t Y_t = Q_0 Y_0 + \int_0^t Q_s dM_s - \int_0^t Q_s A_s ds - \int_0^t Q_s C_s d\langle B \rangle_s$$

and

$$Q_T Y_T = Q_0 Y_0 + \int_0^T Q_s dM_s - \int_0^T Q_s A_s ds - \int_0^T Q_s C_s d\langle B \rangle_s$$

Note that $\int_0^t Q_s dM_s$ is a \mathbb{E}_* -martingale, therefore

$$\begin{aligned} Q_t Y_t &= \mathbb{E}_* \left[Q_T \xi + \int_0^T Q_s A_s ds + \int_0^T Q_s C_s d\langle B \rangle_s | \mathcal{F}_t \right] - \int_0^t Q_s A_s ds - \int_0^t Q_s C_s d\langle B \rangle_s \\ &= \mathbb{E}_* \left[Q_T \xi + \int_t^T Q_s A_s ds + \int_t^T Q_s C_s d\langle B \rangle_s | \mathcal{F}_t \right] \end{aligned}$$

and

$$Y_t = Q_t^{-1} \mathbb{E}_* \left[Q_T \xi + \int_t^T Q_s A_s ds + \int_t^T Q_s C_s d\langle B \rangle_s | \mathcal{F}_t \right].$$

□

Let $(Y_t), (\bar{Y}_t)$ be the unique solutions of the following two one dimensional BSDEs:

$$Y_t = \mathbb{E}_* \left[\xi + \int_t^T g(s, Y_s) ds + \int_t^T f(s, Y_s) d\langle B \rangle_s | \mathcal{F}_t \right] \quad (3.4)$$

$$\bar{Y}_t = \mathbb{E}_* \left[\bar{\xi} + \int_t^T \bar{g}(s, \bar{Y}_s) ds + \int_t^T \bar{f}(s, \bar{Y}_s) d\langle B \rangle_s | \mathcal{F}_t \right] \quad (3.5)$$

We now establish a comparison between them.

Theorem 3.3. *If $\xi \geq \bar{\xi}$, q.s., and for a.e.t, q.s., $\forall y \in \mathbf{R}$, $\delta \geq 0$,*

$$g(t, y + \delta) \geq \bar{g}(t, y) \quad \text{and} \quad f(t, y + \delta) \geq \bar{f}(t, y), \quad (3.6)$$

then we have $\forall t \in [0, T]$, $Y_t \geq \bar{Y}_t$, q.s..

Proof Given any $y_0 \in \mathbf{R}$, we define the following two sequences:

$$Y_t^{i+1} = \mathbb{E}_* \left[\xi + \int_t^T g(s, Y_s^i) ds + \int_t^T f(s, Y_s^i) d\langle B \rangle_s | \mathcal{F}_t \right],$$

$$\bar{Y}_t^{i+1} = \mathbb{E}_* \left[\bar{\xi} + \int_t^T \bar{g}(s, \bar{Y}_s^i) ds + \int_t^T \bar{f}(s, \bar{Y}_s^i) d\langle B \rangle_s | \mathcal{F}_t \right].$$

Set $Y_t^0 = \bar{Y}_t^0 = y_0$. Obviously $\{Y_t^i\}_{i=0}^\infty$ and $\{\bar{Y}_t^i\}_{i=0}^\infty$ are Cauchy sequences in $\mathcal{H}_G^1(0, T; \mathbf{R})$ and $Y_t^i \rightarrow Y_t$, $\bar{Y}_t^i \rightarrow \bar{Y}_t$.

One can check that, by the monotonicity of super linear expectation $\mathbb{E}_*[\cdot | \mathcal{F}_t]$, we have $Y_t^1 \geq \bar{Y}_t^1$, $Y_t^2 \geq \bar{Y}_t^2$, \dots , $Y_t^i \geq \bar{Y}_t^i$, \dots . Thus we get that $Y_t = \lim_{i \rightarrow \infty} Y_t^i \geq \lim_{i \rightarrow \infty} \bar{Y}_t^i = \bar{Y}_t$. \square

4 Probabilistic interpretation for a class of HJB equations

The Hamilton-Jacobi-Bellman (HJB) equation is a second order fully nonlinear partial differential equation which is central to optimal control theory. The solution of the HJB equation is the ‘value function’, which gives the optimal cost-to-go for a given dynamical system with an associated cost function. In general case, the HJB equation does not have a classical (smooth) solution. A notion of generalized solution– viscosity solution has been developed to cover such situations [4, 17]. This section will prove that BSDEs under super/sub linear expectation provide a probabilistic interpretation for the viscosity solution of HJB equations.

Consider the following backward stochastic differential equations under super linear expectation coupled with a forward diffusion driven by a d -dimensional G-Brownian motion with components (B_t^j) **independent** of (B_t^i) , $1 \leq i < j \leq d$:

$$X_s^{t,x} = x + \int_t^s b(X_r^{t,x}) dr + \int_t^s h_j(X_r^{t,x}) d\langle B^j \rangle_r + \int_t^s \sigma_j(X_r^{t,x}) dB_r^j, \quad t \in [0, T], \quad (4.1)$$

$$Y_s^{t,x} = \mathbb{E}_*[\Phi(X_T) + \int_s^T g(X_r^{t,x}, Y_r^{t,x}) dr + \int_s^T f_j(X_r^{t,x}, Y_r^{t,x}) d\langle B^j \rangle_r | \mathcal{F}_s], \quad s \in [t, T], \quad (4.2)$$

where $b, \sigma_j, h_j : \mathbf{R}^n \mapsto \mathbf{R}^n$, $j = 1, \dots, d$, $\Phi : \mathbf{R}^n \mapsto \mathbf{R}$, $g, f_j : \mathbf{R}^n \times \mathbf{R} \mapsto \mathbf{R}$. Here and in the sequence we use the Einstein convention, i.e., the above repeated indices j implies summation.

We assume

(H4.1). $\Phi(\cdot), b(\cdot), h_j(\cdot), \sigma_j(\cdot), g(\cdot, \cdot), f_j(\cdot, \cdot)$, are given uniform Lipschitz functions. i.e., $\forall y^1, y^2$: $|\phi(y^1) - \phi(y^2)| \leq K|y^1 - y^2|$.

By Peng [19, 23] and Theorem 3.1 in this paper, there is a unique pair $(X_s^{t,x}, Y_s^{t,x}) \in \mathcal{M}_G^2(t, T; \mathbf{R}^n) \times \mathcal{H}_G^1(t, T; \mathbf{R})$ for (4.1) and (4.2). It is not difficult to obtain the following estimates. Most of proofs can be founded in Peng [23] Ch.V, Sec. 3.

Lemma 4.1. (i) $\mathbb{E}^* \left| X_s^{t,x} - X_s^{t,x'} \right| \leq C|x - x'|$, $s \in [t, T]$.

(ii) $\mathbb{E}^* |X_s^{t,x}|^p \leq C(1 + |x|^p)\delta^{\frac{p}{2}}$, $p \geq 2$, $s \in [t, T]$.

(iii) $\mathbb{E}^* |X_{t+\delta}^{t,x} - x|^2 \leq C(1 + |x|^2)\delta$, $\delta \in [0, T - t]$.

(iv) $|Y_t^{t,x} - Y_t^{t,x'}| \leq C|x - x'|$.

(v) $|Y_t^{t,x}| \leq C(1 + |x|)$.

(vi) $|Y_{t+\delta}^{t+\delta,x} - Y_t^{t,x}| \leq C(1 + |x|)(\delta^{\frac{1}{2}} + \delta)$.

(vii) $\mathbb{E}^* |Y_{t+\delta}^{t,x} - Y_t^{t,x}| \leq C(1 + |x|)\delta$.

where the constant C dose not depend on the (t, x, δ) .

Note that a nonlinear Feynman-Kac formula has been established in [23], where the control variable in associated PDE is a matrix. Here in the framework of superlinear expectation, we assume some independence among components of G-Brownian motion and obtain an obvious form of HJB equation with vector-valued control variable. It is seen that in the next section results here give some natural applications to stochastic control and uncertainty volatility model.

We define

$$u(t, x) := Y_t^{t,x} = \mathbb{E}_* \left[\Phi(X_T^{t,x}) + \int_t^T g(X_r^{t,x}, Y_r^{t,x}) dr + \int_t^T f_j(X_r^{t,x}, Y_r^{t,x}) d\langle B^j \rangle_r | \mathcal{F}_t \right].$$

Since $(X_s^{t,x}, Y_s^{t,x})$ is dependant of \mathcal{F}_t , $u(t, x)$ is a deterministic function.

Theorem 4.1. $u(t, x)$ is a viscosity solution of the following HJB equation

$$\partial_t u + \inf_{\alpha \in \Gamma} \{ \mathcal{L}(x, \alpha) u + g(x, u) \} = 0 \quad (4.3)$$

$$u|_{t=T} = \Phi. \quad (4.4)$$

where the control variable α is selected dynamically within Γ . $\mathcal{L}(x, \alpha)$ is a second order elliptic partial differential operator parameterized by α ,

$$\mathcal{L}(x, \alpha) = \frac{1}{2} \sum_{\mu, \nu=1}^n \left(\sum_{j=1}^d \sigma_{\mu j} \sigma_{\nu j}(x) \alpha_j^2 \right) \cdot \partial_{x^\mu x^\nu} + \sum_{i=1}^n \left(b_i(x) + \sum_{j=1}^d h_{ij}(x) \alpha_j^2 \right) \partial_{x^i} + \sum_{j=1}^d f_j(x, u) \alpha_j^2.$$

Proof (vi) and (iv) in Proposition 4.1 lead to the continuity of $u(t, x)$ in (t, x) . Now we prove $u(t, x)$ is a viscosity of PDE (4.3). By the definition of $u(t, x)$, for $\delta \in (0, T - t)$, we have,

$$\begin{aligned} u(t + \delta, X_{t+\delta}^{t,x}) &= Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t,x}} = Y_{t+\delta}^{t,x} \\ &= \mathbb{E}_* \left[\Phi(X_T^{t,x}) + \int_{t+\delta}^T g(X_r^{t,x}, Y_r^{t,x}) dr + \int_{t+\delta}^T f_j(X_r^{t,x}, Y_r^{t,x}) d\langle B^j \rangle_r | \mathcal{F}_{t+\delta} \right]. \end{aligned}$$

Thus

$$u(t, x) = \mathbb{E}_* \left[u(t + \delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} g(X_r^{t,x}, Y_r^{t,x}) dr + \int_t^{t+\delta} f_j(X_r^{t,x}, Y_r^{t,x}) d\langle B^j \rangle_r \right].$$

Now for fixed $(t, x) \in (0, T) \times \mathbf{R}^n$, Let $\psi \in C_b^{1,3}([0, T] \times \mathbf{R}^n)$ s.t $\psi \geq u$ and $\psi(t, x) = u(t, x)$. By G-Itô formula, it follows that, for $\delta \in (0, T - t)$,

$$0 \leq \mathbb{E}_* \left[\psi(t + \delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} g(X_r^{t,x}, Y_r^{t,x}) dr + \int_t^{t+\delta} f_j(X_r^{t,x}, Y_r^{t,x}) d\langle B^j \rangle_r - \psi(t, x) \right] = \mathbb{E}_* [\mathbf{I}_\delta^1]$$

Where we denote

$$\begin{aligned} \mathbf{I}_\delta^1 &= \int_t^{t+\delta} \partial_{\psi_t}(r, X_r^{t,x}) dr + \int_t^{t+\delta} \partial_{x^i} \psi(r, X_r^{t,x}) (b_i(X_r^{t,x}) dr + h_{ij}(X_r^{t,x}) d\langle B^j \rangle_r) \\ &\quad + \frac{1}{2} \int_t^{t+\delta} \partial_{x^\mu x^\nu} \psi(r, X_r^{t,x}) \sigma_{\mu j} \sigma_{\nu j}(X_r^{t,x}) d\langle B \rangle_r^j \\ &\quad + \int_t^{t+\delta} g(X_r^{t,x}, Y_r^{t,x}) dr + \int_t^{t+\delta} f_j(X_r^{t,x}, Y_r^{t,x}) d\langle B^j \rangle_r, \end{aligned}$$

and

$$\begin{aligned} \mathbf{I}_\delta^2 &= \delta [\partial_{\psi_t}(t, x) + \partial_{x^i} \psi(t, x) b_i(x) + g(x, u(t, x))] \\ &\quad + \left[\frac{1}{2} \partial_{x^\mu x^\nu} \psi(t, x) \sigma_{\mu j} \sigma_{\nu j}(x) + h_{ij}(x) + f_j(x, u(t, x)) \right] (\langle B^j \rangle_{t+\delta} - \langle B^j \rangle_t). \end{aligned}$$

In the following we denote C as a universal constant independent of δ . Then using the Lipschitz conditions and by Lemma 4.1 we have

$$\begin{aligned} |\mathbb{E}_* [\mathbf{I}_\delta^1] - \mathbb{E}_* [\mathbf{I}_\delta^2]| &\leq \mathbb{E}_* |\mathbf{I}_\delta^1 - \mathbf{I}_\delta^2| \\ &\leq C \mathbb{E}_* \left[\int_t^{t+\delta} |Y_r^{t,x} - Y_t^{t,x}| dr + \int_t^{t+\delta} |X_r^{t,x} - x| dr \right. \\ &\quad \left. + \int_t^{t+\delta} (|X_r^{t,x}| + |x|) |X_r^{t,x} - x| dr + \int_t^{t+\delta} |X_r^{t,x}|^2 |X_r^{t,x} - x| dr \right] \\ &\leq C(1 + |x|)\delta^2 + C(1 + |x| + |x|^2 + |x|^3)\delta^{\frac{3}{2}} \end{aligned}$$

We denote $R_\delta := C(1 + |x|)\delta + C(1 + |x| + |x|^2 + |x|^3)\delta^{\frac{1}{2}}$. Then we obtain

$$\begin{aligned} 0 &\leq \frac{1}{\delta} \mathbb{E}_* [\mathbf{I}_\delta^1] \leq \frac{1}{\delta} \mathbb{E}_* [\mathbf{I}_\delta^2] + R_\delta \\ &= \frac{1}{\delta} \mathbb{E}_* [\partial_{\psi_t}(t, x) \delta + \partial_{x^i} \psi(t, x) \cdot b_i(x) \delta + g(x, u) \delta \\ &\quad + \left(\frac{1}{2} \partial_{x^\mu x^\nu} \psi(r, x) \sigma_{\mu j} \sigma_{\nu j}(x) + \partial_{x^i} \psi(t, x) \cdot h_{ij}(x) + f_j(x, u) \right) \cdot (\langle B^j \rangle_{t+\delta} - \langle B^j \rangle_t)] + R_\delta \\ &= \partial_t u + \inf_{\alpha \in \Gamma} \{ \mathcal{L}(x, \alpha) u + g(x, u) \} + R_\delta \end{aligned}$$

Let $\delta \rightarrow 0$, we deduce that $u(t, x)$ is a viscosity subsolution of PDE (4.3). Similarly we can prove that $u(t, x)$ is also a viscosity supersolution of PDE (4.3). \square

The HJB equation is the infinitesimal version of the dynamic programming principle: it describes the local behavior of the super linear expectation. The HJB equation is also called dynamic programming equation. Theorem 4.1 provides an immediate proof of the dynamic programming principle.

Corollary 4.1. *Let u be the solution of PDE (4.3). Then for every $\delta \in [0, T - t]$, we have*

$$u(t, x) = \inf_{P \in \mathcal{P}} E_P \left[u(t + \delta, X_{t+\delta}^{t,x}) + \int_t^{t+\delta} g(X_r^{t,x}, Y_r^{t,x}) dr + \int_t^{t+\delta} f_j(X_r^{t,x}, Y_r^{t,x}) d\langle B^j \rangle_r \right].$$

Corollary 4.2. *Let $(W_t), (B_t), (Z_t)$ be three G -Brownian motion with uncertainty set $\Gamma_i, i = 1, 2, 3$, with (W_t) independent of (B_t) and (Z_t) , (B_t) independent of (Z_t) . Then the following system*

$$X_s = x + \int_t^s b(X_r) dr + \int_t^s h_j(X_r) d\langle W^j \rangle_r + \int_t^s \sigma_j(X_r) dB_r^j \quad (4.5)$$

$$Y_s = \mathbb{E}_* \left[\Phi(X_T) + \int_s^T g(X_r, Y_r) dr + \int_s^T f_j(X_r, Y_r) d\langle Z^j \rangle_r | \mathcal{F}_s \right], \quad (4.6)$$

provide a probabilistic interpretation for the viscosity solution of following equation

$$\begin{aligned} & \partial_t u + \inf_{\alpha \in \Gamma_1} \left\{ \sum_{i=1}^n \sum_{j=1}^d h_{ij}(x) \alpha_j^2 \partial_{x^i} u \right\} + \inf_{\alpha \in \Gamma_2} \left\{ \frac{1}{2} \sum_{\mu, \nu=1}^n \left(\sum_{j=1}^d \sigma_{\mu j} \sigma_{\nu j}(x) \alpha_j^2 \right) \partial_{x^\mu x^\nu} u \right\} \\ & + \inf_{\alpha \in \Gamma_3} \left\{ \sum_{j=1}^d f_j(x, u) \alpha_j^2 \right\} + \sum_{i=1}^n b_i(x) \partial_{x^i} u + g(t, x, u) = 0 \end{aligned} \quad (4.7)$$

$$u|_{t=T} = \Phi. \quad (4.8)$$

As a converse of Theorem 4.1, we have

Theorem 4.2. *If PDE (4.3) has a classical solution $u(t, x) \in C_b^{1,2}$, then $u(s, X_s^{t,x})$ solves BSDE (4.2).*

Proof We denote $N_s := - \int_t^s \inf_{\alpha \in \Gamma} (h_{ij} \partial_{x^i} u + \frac{1}{2} \partial_{x^\mu x^\nu} u \sigma_{\mu j} \sigma_{\nu j} + f_j) \alpha_j^2 dr + \int_t^s (h_{ij} \partial_{x^i} u + \frac{1}{2} \partial_{x^\mu x^\nu} u \sigma_{\mu j} \sigma_{\nu j} + f_j) d\langle B^j \rangle_r$. Here we use the Einstein convention, i.e., the repeated indices μ, ν, i, j imply summation. Then by G-Itô formula, we have

$$\begin{aligned} u(s, X_s^{t,x}) &= u(t, x) + \int_t^s \partial_{x^i} u (b_i dr + h_{ij} d\langle B^j \rangle_r + \sigma_{ij} dB_r^j) + \frac{1}{2} \int_t^s \partial_{x^\mu x^\nu} u \sigma_{\mu j} \sigma_{\nu j} d\langle B^j \rangle_r \\ &= u(t, x) + \int_t^s \sigma_{ij} \partial_{x^i} u dB_r^j + N_s - \int_t^s g dr - \int_t^s f_j d\langle B^j \rangle_r \end{aligned}$$

and

$$\Phi(X_T) = u(s, X_T^{t,x}) = u(t, x) + \int_t^T \sigma_{ij} \partial_{x^i} u dB_r^j + N_T - \int_t^T g dr - \int_t^T f_j d\langle B^j \rangle_r.$$

Observe that (N_s) is a \mathbb{E}_* -martingale, thus

$$\begin{aligned} u(s, X_s^{t,x}) &= \mathbb{E}_* \left[\Phi(X_T) + \int_t^T g(X_r^{t,x}, u(r, X_r^{t,x})) dr + \int_t^T f_j(X_r^{t,x}, u(r, X_r^{t,x})) d\langle B^j \rangle_r \middle| \mathcal{F}_s \right] \\ &\quad - \int_t^s g(X_r, Y_r) dr - \int_t^s f_j(X_r, Y_r) d\langle B^j \rangle_r \\ &= \mathbb{E}_* \left[\Phi(X_T) + \int_s^T g(X_r^{t,x}, u(r, X_r^{t,x})) dr + \int_s^T f_j(X_r^{t,x}, u(r, X_r^{t,x})) d\langle B^j \rangle_r \middle| \mathcal{F}_s \right]. \end{aligned}$$

Therefore $Y_s := u(s, X_s^{t,x})$ is the unique solution of BSDE (4.2). \square

5 Applications

With the Feynman-Kac formula, we now show some applications of G-expectation(G-BSDE). In fact the condition of quasi-continuity is not necessary for random variables and processes in section 3 and 4 [23, 10]. So we can apply all the results in section 3 and 4 to practical problems not in quasi-continuous spaces.

5.1 Connection with stochastic control problem

Let $(\Omega, (\mathcal{F}_t), E)$ be a space of linear expectation, $(W)_t$ is a d -dimensional standard E -Brownian motion. $(\mathcal{F}_t)_{t \geq 0}$ is the usual augmented Brownian motion filtration. Now consider the following stochastic control problem with n -dimensional state process

$$x_s = x + \int_0^s (b(x_r) + h_j(x_r) \alpha_j^2(r)) dr + \int_0^s \sigma_j(x_r) \alpha_j^2(r) dW_r^j \quad (5.1)$$

where $(\alpha_s)_{s \geq 0} \in \mathcal{A}$, a set of \mathcal{F}_s -adapted processes taking values in a compact set $A \subseteq \mathbf{R}^d$, called control process. (x_s) is called the trajectory corresponding to (α_s) . Here we still use the Einstein convention. For any given $t \in [0, T]$, we introduce the following one dimensional BSDE:

$$y_s = E \left[\Phi(x_{T-t}) + \int_s^{T-t} (g(x_r, y_r) + f_j(x_r, y_r) \alpha_j^2(r)) dr \middle| \mathcal{F}_s \right], \quad s \in [0, T-t] \quad (5.2)$$

We assume

(H5.1). b, h and σ are continuously differentiable in x , their derivatives b_x, h_x, σ_x being bounded.

(H5.2). f, g are continuously differentiable in (x, y) , their derivatives f_x, f_y, g_x, g_y being bounded.

(H5.3). $\Phi(x)$ is a uniform Lipschitz function.

Under conditions **(H5.1)**~**(H5.3)**, the system (5.1) and (5.2) is well defined and there is a unique pair (x_s, y_s) solving (5.1) and (5.2) (see [17]). Then we can define the so called cost function as

$$J_{x,t}(\alpha(\cdot)) = y_0 (= Ey_0)$$

The value function of this optimal control problem is defined by

$$V(x, T-t; \alpha(\cdot)) = \inf_{\alpha \in \mathcal{A}} J(x, t; \alpha(\cdot), \Phi(\cdot))$$

By Peng [17], we have

Proposition 5.1. *Let (H5.1)~(H5.3) hold. Then for fixed Φ , the value function $u(t, x) := V(x, T - t; \Phi(\cdot))$, $(t, x) \in [0, T] \times \mathbf{R}^n$ is a viscosity solution of HJB equation (4.3) with control domain $A \subseteq \mathbf{R}^d$.*

Then by the uniqueness of viscosity solution for PDE(4.3) (see Ishii [8]), we have

Theorem 5.1. *Let (H5.1)~(H5.3) hold and $A = \Gamma$. Then $V(x, T - t; \Phi(\cdot)) = u(t, x) = Y_t^{t, x}$.*

A G-BSDE is in fact a recursive super (sub) strategy. Theorem 5.1 establishes a direct connection between general recursive sub strategies and stochastic control problems.

5.2 The uncertainty volatility model

The uncertainty volatility model (UVM) for pricing and hedging derivative securities in an environment where the volatility is not known precisely, but is assumed instead to lie between two non-negative extreme values $\underline{\sigma}$ and $\bar{\sigma}$. Let us denote \mathcal{P} the class of all probability measures P on the set of path $\{S_t, t \in [0, T]\}$ where (S_t) is the price of a stock. For simplicity we restrict our discussion to derivative securities based on a single liquidly traded stock which pays no dividends over the contract's lifetime. Denote $\mathbb{E}_*[\cdot] = \inf_{P \in \mathcal{P}} E_P[\cdot]$, $\mathbb{E}^*[\cdot] = \sup_{P \in \mathcal{P}} E_P[\cdot]$. Let $(B_t)_{t \geq 0}$ be a one-dimensional Brownian motion under \mathbb{E}^* and \mathbb{E}_* . $(\mathcal{F}_t)_{t \geq 0}$ is the usual augmented Brownian motion filtration. We denote parameters $\bar{\sigma}^2 = \mathbb{E}^*[\langle B \rangle_{t=1}]$, $\underline{\sigma}^2 = \mathbb{E}_*[\langle B \rangle_{t=1}]$. If there is no arbitrage, the forward stock price should satisfy the risk-neutral Itô equation:

$$dS_u = rS_u du + S_u dB_u, u \in [t, T] \quad (5.3)$$

$$S_t = x \quad (5.4)$$

where r is the riskless interest rate.

Assume that at a given maturity date T , a derivative security is characterized by $\Phi(S_T)$, where $\Phi(\cdot)$ is a known function of the price of the underlying stock. Then the offer price and bid price of this derivative should satisfy

$$\overline{Y}_s^{t, x} = \mathbb{E}^* \left[\Phi(S_T) + \int_s^T (-r \overline{Y}_u^{t, x}) du | \mathcal{F}_s \right], s \in [t, T] \quad (5.5)$$

$$\underline{Y}_s^{t, x} = \mathbb{E}_* \left[\Phi(S_T) + \int_s^T (-r \underline{Y}_u^{t, x}) du | \mathcal{F}_s \right], s \in [t, T] \quad (5.6)$$

However in Avellaneda, Levy and Parás [1] or by Theorem 4.1, the offer price and bid price are characterized as $w^*(t, S_t)$ and $w_*(t, S_t)$ respectively:

$$\frac{\partial w^*}{\partial t}(t, x) + r(x \frac{\partial w^*}{\partial x}(t, x) - w^*(t, x)) + \frac{x^2}{2} \sup_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} \left\{ \sigma^2 \frac{\partial^2 w^*}{\partial x^2}(t, x) \right\} = 0 \quad (5.7)$$

$$w^*|_{t=T} = \Phi. \quad (5.8)$$

and

$$\frac{\partial w_*}{\partial t}(t, x) + r(x \frac{\partial w_*}{\partial x}(t, x) - w_*(t, x)) + \frac{x^2}{2} \inf_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} \left\{ \sigma^2 \frac{\partial^2 w_*}{\partial x^2}(t, x) \right\} = 0 \quad (5.9)$$

$$w_*|_{t=T} = \Phi. \quad (5.10)$$

which are referred as the Black-Scholes-Barrenblett (BSB) equations. They are a generalization of the classical Black-Scholes PDE, and reduce to it in the case of $\underline{\sigma} = \bar{\sigma}$.

By results in section 4, we have

Proposition 5.2. *The two characterization for the offer price and bid price are equivalent, i.e.:*

$$w^*(t, x) = \overline{Y}_t^{t, x} = \sup_P E_P[e^{-r(T-t)} \Phi(X_T)]$$

$$w_*(t, x) = \underline{Y}_t^{t, x} = \inf_P E_P[e^{-r(T-t)} \Phi(X_T)]$$

From the above we see that BSDEs under sub (super) linear expectation provide a new characterization for the UVM model. Another problem is to solve either BSDE(5.5) and (5.6) or BSB(5.7) and (5.9). When Φ is convex, the BSB prices coincide the Black-Scholes prices at volatility $\bar{\sigma}$ and $\underline{\sigma}$. Generally, we can not find explicit solutions for the BSB equations. We now provide a representation for solutions of the Black-Scholes-Barrenblett equations.

Theorem 5.2.

$$w^*(t, x) = e^{-r(T-t)} \sup_{\underline{\sigma} \leq \alpha_u \leq \bar{\sigma}} E \left[\Phi(xe^{r(T-t) + \int_0^{T-t} \alpha_u dW_u - \frac{1}{2} \int_0^s |\alpha_u|^2 du}) \right],$$

$$w_*(t, x) = e^{-r(T-t)} \inf_{\underline{\sigma} \leq \alpha_u \leq \bar{\sigma}} E \left[\Phi(xe^{r(T-t) + \int_0^{T-t} \alpha_u dW_u - \frac{1}{2} \int_0^s |\alpha_u|^2 du}) \right].$$

Proof Note that $x_s = x \exp\{rs + \int_0^s \alpha_u dW_u - \frac{1}{2} \int_0^s |\alpha_u|^2 du\}$, $y_s = e^{-r(T-t)} E[\Phi(x_{T-t}) | \mathcal{F}_s]$ solves

$$x_s = x + \int_0^s r x_u du + \int_0^s \alpha_u x_u dW_u, \quad (5.11)$$

$$y_s = E \left[\Phi(x_{T-t}) + \int_s^{T-t} (-r y_u) du | \mathcal{F}_s \right], \quad s \in [0, T-t], \quad (5.12)$$

where (α_t) is the control process between $\underline{\sigma}$ and $\bar{\sigma}$. (W_t) is a standard Brownian motion under linear expectation E . By Proposition 5.1, we have

$$\begin{aligned} w_*(t, x) &= \inf_{\underline{\sigma} \leq \alpha_u \leq \bar{\sigma}} E y_0 = e^{-r(T-t)} \inf_{\underline{\sigma} \leq \alpha_u \leq \bar{\sigma}} E [\Phi(x_{T-t})] \\ &= e^{-r(T-t)} \inf_{\underline{\sigma} \leq \alpha_u \leq \bar{\sigma}} E \left[\Phi(xe^{r(T-t) + \int_0^{T-t} \alpha_u dW_u - \frac{1}{2} \int_0^s |\alpha_u|^2 du}) \right] \end{aligned}$$

By Proposition 5.3, we have

$$\begin{aligned} w^*(t, x) &= \overline{Y}_t^{t, x} = \sup_P E_P[e^{-r(T-t)} \Phi(X_T)] = -\inf_P E_P[-e^{-r(T-t)} \Phi(X_T)] = -\underline{Y}_t^{t, x}(-\Phi) \\ &= e^{-r(T-t)} \sup_{\underline{\sigma} \leq \alpha_u \leq \bar{\sigma}} E \left[\Phi(xe^{r(T-t) + \int_0^{T-t} \alpha_u dW_u - \frac{1}{2} \int_0^s |\alpha_u|^2 du}) \right]. \end{aligned}$$

□

5.3 Representation for solutions of G-heat equations

We can use the following two PDEs

$$\partial_t u(t, x) + G^*(D_x^2 u) = 0, \quad u(T, x) = \Phi(x), \quad (5.13)$$

$$\partial_t u(t, x) + G_*(D_x^2 u) = 0, \quad u(T, x) = \Phi(x), \quad (5.14)$$

to introduce G-expectation ([19]), where $G^*(a) = \frac{1}{2} \sup_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} \{\sigma^2 a\}$, $G_*(a) = \frac{1}{2} \inf_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} \{\sigma^2 a\}$. So PDE(5.13) and (5.14) are called G-heat equations. It is fundamentally important to solve G-heat equations in the theory of G-expectation. Hu [6] constructed explicit solutions of G*-heat equation (5.13) with a class of terminal condition $\Phi(x) = x^n$ for each integer $n \geq 1$ and in [5] a representation for the solution of (5.13) is given on the Wiener space. We now provide a representation for solutions of the above G-heat equations (5.14) with terminal condition $\Phi \in C_{Lip}(\mathbf{R})$ under a given space of linear expectation. In fact results in this subsection also holds for $\Phi \in C_{l,Lip}(\mathbf{R})$. The proof is a natural sequence of Proposition 5.1 and Theorem 4.1. For simplicity we only consider spatial variable $x \in \mathbf{R}$. Similarly to Theorem 5.2, we have

Theorem 5.3.

$$u_*(t, x) = \inf_{\underline{\sigma} \leq \alpha \leq \bar{\sigma}} E \left[\Phi \left(x + \int_0^{T-t} \alpha_r dW_r \right) \right]$$

solves PDE(5.14).

Proof Let (W_t) be a standard Brownian motion under a linear expectation E . Consider

$$\begin{aligned} x_s &= x + \int_0^s \alpha_r dW_r \\ y_s &= E[\Phi(x_{T-t}) | \mathcal{F}_s], \quad s \in [0, T-t]. \end{aligned}$$

By Proposition 5.1, we have

$$u_*(t, x) = \inf_{\underline{\sigma} \leq \alpha \leq \bar{\sigma}} E[\Phi(x_{T-t})] = \inf_{\underline{\sigma} \leq \alpha \leq \bar{\sigma}} E \left[\Phi \left(x + \int_0^{T-t} \alpha_r dW_r \right) \right].$$

□

Let (B_t) be a G-Brownian motion under \mathbb{E}_* , X be a \mathbb{E}_* -normal distributed random variable.

Corollary 5.1. Assuming further $\Phi \in C^2(\mathbf{R})$, We have

(i)

$$\begin{aligned} \mathbb{E}_*[\Phi(x + (B_T - B_t))] &= \inf_{\underline{\sigma} \leq \alpha \leq \bar{\sigma}} E \left[\Phi \left(x + \int_0^{T-t} \alpha_r dW_r \right) \right] \\ &= \Phi(x) + \frac{1}{2} \inf_{\underline{\sigma} \leq \alpha \leq \bar{\sigma}} E \left[\int_0^{T-t} \Phi'' \left(x + \int_0^r \alpha_u dW_u \right) \alpha_r^2 dr \right] \end{aligned}$$

(ii)

$$\begin{aligned}\mathbb{E}_*[\Phi(X)] &= \inf_{\underline{\sigma} \leq \alpha, \leq \bar{\sigma}} E[\Phi(\int_0^1 \alpha_r dW_r)] \\ &= \Phi(0) + \frac{1}{2} \inf_{\underline{\sigma} \leq \alpha, \leq \bar{\sigma}} E \left[\int_0^1 \Phi''(\int_0^r \alpha_u dW_u) \alpha_r^2 dr \right]\end{aligned}$$

Proof Consider

$$\begin{aligned}X_s &= x + \int_t^s dB_r \\ Y_s &= \mathbb{E}_* [\Phi(X_T) | \mathcal{F}_s], \quad s \in [t, T].\end{aligned}$$

By Theorem 5.1, we have $u(t, x) = Y_t^{t, x} = \mathbb{E}_* [\Phi(X_T)] = \mathbb{E}_* [\Phi((x + (B_T - B_t)))]$ and $u(t, x) = V(x, T - t; \Phi(\cdot)) = \inf_{\underline{\sigma} \leq \alpha, \leq \bar{\sigma}} E[\Phi(x + \int_0^{T-t} \alpha_r dW_r)]$. So we obtain the desired results. \square

Remark 5.1. When Φ is a convex function,

$$\mathbb{E}_*[\Phi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\underline{\sigma}y) e^{-\frac{y^2}{2}} dy.$$

and

$$\mathbb{E}^*[\Phi(X)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \Phi(\bar{\sigma}y) e^{-\frac{y^2}{2}} dy,$$

But if Φ is a concave function, $\underline{\sigma}, \bar{\sigma}$ must interchange their positions. From the above we see that there is no fixed density function for G -normal distributed random variable X in the traditional sense. Thus many problems under sub(super)-linear expectation can not be solved via methods of density function.

6 Appendix: Discussion on the dominated convergence theorem

The theorem of dominated convergence is fundamentally important in the theory of classical probability. We initially want to embed a control process $\alpha_t = \frac{d\langle B \rangle_t}{dt}$ into the coefficient g of BSDE (4.2). However when we derive the associated HJB equation, we find that the dominated convergence theorem does not hold in general under sublinear expectation induced by mutually singular probability measures.. We now give a sufficient condition and a counterexample about it.

Lemma 6.1. Assume that a sequence X_n converges to X in the sense of ‘capacity’, i.e.: for any $\varepsilon > 0$,

$$C^*(|X_n - X| > \varepsilon) \rightarrow 0.$$

Then there is a subsequence $\{X_{n_k}\}$ such that $X_{n_k} \rightarrow X$, q.s..

Proof Since $X_n \rightarrow X$ in the sense of ‘capacity’, we can find a subsequence $\{X_{n_k}\}$ such that

$$C^*(|X_{n_k} - X| > \frac{1}{k}) \leq \frac{1}{k^2}.$$

Then by the Borel-Cantelli Lemma (see Peng [23] Ch.VI, Lemma 1.5), we deduce that

$$C^*(\overline{\lim}_{k \rightarrow \infty} \{|X_{n_k} - X| > \frac{1}{k}\}) = 0.$$

Obviously X_{n_k} converges to X on $(\overline{\lim}_{k \rightarrow \infty} \{|X_{n_k} - X| > \frac{1}{k}\})^c$. \square

Let $L_b^1(\Omega)$ be the completion of all bounded $\mathcal{B}(\Omega)$ -measurable functions under norm $\|X\|_1 = \mathbb{E}^*|X|$

Proposition 6.1. *Let $\{X_n\}_{n=1}^\infty$ be a measurable sequence on (Ω, \mathcal{F}) such that $|X_n| \leq Y$, q.s., $n=1,2,\dots$ and $Y \in L_b^1(\Omega)$. If $X_n \rightarrow X$ in the sense of ‘capacity’, then $\lim_{n \rightarrow \infty} \mathbb{E}^*[X_n] = \mathbb{E}^*[X]$.*

Proof By Lemma 6.1, there is a subsequence $\{X_{n_k}\}$ such that $X_{n_k} \rightarrow X$, q.s.. Therefore $|X| \leq Y$, q.s. and $X \in L_b^1(\Omega)$. For any $\varepsilon > 0$, we denote $A_n := \{|X_n - X| \geq \frac{\varepsilon}{4}\}$. Then

$$\mathbb{E}^*[|X_n - X| \cdot \mathbf{1}_{\Omega/A_n}] \leq \frac{\varepsilon}{4} C^*(\Omega/A_n) < \frac{\varepsilon}{2}. \quad (6.1)$$

By the complete continuity of \mathbb{E}^* on $L_b^1(\Omega)$ (see Peng [23] Ch.VI, Proposition 1.19), there exists a $\delta > 0$, for any $A \subset \Omega$, $C^*(A) < \delta$ such that $\mathbb{E}^*[Y \mathbf{1}_A] < \frac{\varepsilon}{4}$. For this δ , there exists n , when $n \geq N$, $C^*(A_n) < \delta$. Therefore

$$\mathbb{E}^*[|X_n - X| \cdot \mathbf{1}_{A_n}] \leq 2\mathbb{E}^*[Y \mathbf{1}_{A_n}] < \frac{\varepsilon}{2}. \quad (6.2)$$

(6.1) and (6.2) lead to that

$$|\mathbb{E}^*X_n - \mathbb{E}^*X| \leq \mathbb{E}^*[|X_n - X|] \leq \mathbb{E}^*[|X_n - X| \cdot \mathbf{1}_{\Omega/A_n}] + \mathbb{E}^*[|X_n - X| \cdot \mathbf{1}_{A_n}] < \varepsilon, \quad n \geq N.$$

The proof is completed. \square

‘q.s.’ convergence does not implies ‘capacity’ convergence. If we replace ‘capacity’ convergence by ‘q.s.’ convergence in Proposition 6.1, the dominated convergence theorem does not hold true.

Counterexample 6.1. *Let (B_t) be a one dimensional $G_{[\underline{\sigma}^2, \overline{\sigma}^2]}$ -Brownian motion with $\underline{\sigma}^2 < \overline{\sigma}^2$. For fixed t , we denote $X_\delta^t := \frac{\langle B \rangle_{t+\delta} - \langle B \rangle_t}{\delta} - \frac{\langle B \rangle_t - \langle B \rangle_{t-\delta}}{\delta}$, $0 < \delta < t$. Obviously $X_\delta^t \rightarrow 0$, q.s. when $\delta \downarrow 0$ and $\mathbb{E}^*[\lim_{\delta \downarrow 0} X_\delta^t] = 0$. However since $\frac{\langle B \rangle_{t+\delta} - \langle B \rangle_t}{\delta}$ is independent of $\frac{\langle B \rangle_t - \langle B \rangle_{t-\delta}}{\delta}$, we have that*

$$\begin{aligned} \lim_{\delta \downarrow 0} \mathbb{E}^*[X_\delta^t] &= \lim_{\delta \downarrow 0} \mathbb{E}^* \left(\frac{\langle B \rangle_{t+\delta} - \langle B \rangle_t}{\delta} - \frac{\langle B \rangle_t - \langle B \rangle_{t-\delta}}{\delta} \right) \\ &= \lim_{\delta \downarrow 0} \mathbb{E}^* \left[\mathbb{E}^* \left(\frac{\langle B \rangle_{t+\delta} - \langle B \rangle_t}{\delta} - y \right)_{y = \frac{\langle B \rangle_t - \langle B \rangle_{t-\delta}}{\delta}} \right] \\ &= \lim_{\delta \downarrow 0} \mathbb{E}^* \left[\overline{\sigma}^2 - \frac{\langle B \rangle_t - \langle B \rangle_{t-\delta}}{\delta} \right] \\ &= \overline{\sigma}^2 - \underline{\sigma}^2 > 0. \end{aligned}$$

From the above we see that $\lim_{\delta \downarrow 0} \mathbb{E}^*[X_\delta^t] \neq \mathbb{E}^*[\lim_{\delta \downarrow 0} X_\delta^t]$. \square

References

- [1] M. Avellaneda, A. Levy and A. Paras, *Pricing and hedging derivative securities in markets with uncertain volatilities*, Appl Math Finance, 2 (1995), 73–88.
- [2] F. Q. Gao, *Pathwise properties and homeomorphic flows for stochastic differential equations driven by G-Brownian motion*, Stochastic Processes and their Applications, 119 (2009), 3356–3382.

- [3] P. Cheridito, H. M. Soner, N. Touzi, and N. Victoir, *Second order BSDE's and fully nonlinear PDEs*, Communications in Pure and Applied Mathematics, 60 (2007), 1081-1110.
- [4] M. Crandall, H. Ishii and P. L. Lions, *User's guide to viscosity solutions of second order partial differential equations*, Bull Amer Math Soc, 27 (1992), 1-67.
- [5] L. Denis, M. Hu and S. Peng, *Function spaces and capacity related to a sublinear expectation: application to G-Brownian motion pathes*, arXiv:0802.1240 v1 [math.PR] 9 Feb, 2008.
- [6] M. Hu, *Explicit solutions of G-heat equation with a class of initial conditions by G-Brownian motion*, arXiv:0907.2748 v1 [math.PR] 16 Jul 2009.
- [7] M. Hu and S. Peng, *On representation theorem of G-expectations and paths of G-Brownian motion*, Acta Math Appl Sinica English Series, 25 (2009), 1-8.
- [8] H. Ishii, *On uniqueness and existence of viscosity solutions of fully nonlinear second order PDEs*, Comm. Pure Appl. Math., 42 (1989), 15-45.
- [9] M. Kac, *On some connections between probability theory and differential and integral equations*, Proc. 2nd Berkley Symp. on Math. Stat. and Prob. Univ. of California Press, Berley and Los Angeles, (1951)
- [10] X. Li and S. Peng, *Stopping Times and Related Ito's Calculus with G-Brownian Motion*, arXiv:0910.3871 v1 [math.PR] 20 Oct 2009.
- [11] Y. Hu and J. Ma, *Nonlinear Feynman-Kac formula and discrete-functional-type BSDEs with continuous coefficients*, Stochastic Processes and their Applications, 112 (2004), 23 - 51.
- [12] B. Øksendal, *Stochastic Differential Equation*, 6nd ed. Springer-Verlag, 2005.
- [13] E. Pardoux and S. Peng, *Backward stochastic differential equations and quasilinear parabolic partial differential equations*, In: Rozuvskii, B. L., Sowers, R. B. (eds.) Stochastic partial differential equations and their applications, Lect. Notes Control Inf. Sci, vol. 176, Berlin Heidelberg New York: Springer, 1992, 200-217.
- [14] E. Pardoux and S. Peng, *Backward doubly stochastic differential equations and systems of quasilinear SPDEs*, Probab. Theory Relat. Fields, 98 (1994), 209-227.
- [15] S. Peng, *Probabilistic interpretation for systems of quasilinear parabolic partial differential equations*, Stochstics, 37 (1991), 61-74.
- [16] S. Peng, *A non linear Feynman-Kac formula and applications*, In: Chen, S. P., Yong, J. M. (eds.) Proc. of Symposium on system science and control theory, Singapore: World Scientific, 1992, 173-184.
- [17] S. Peng, *A generalized dynamic programming priciples and Hamilton-Jacobi-Bellman equations*, Stochastics and Stochastics reports, 38 (1992), 119-134,
- [18] S. Peng, *Nonlinear expectations and nonlinear Markov chains*, Chinese Ann Math Ser B, 26 (2005), 159-184.
- [19] S. Peng, *G-expectation, G-Brownian motion and related stochastic calculus of Itos type*, In: Stochastic Analysis and Applications, The Abel Symposium 2005, Abel Symposia 2. New York: Springer-Verlag, 2006, 541-567.
- [20] S. Peng, *G-Brownian motion and dynamic risk measure under volatility uncertainty*, arXiv:0711.2834 v1 [math.PR] 19 Nov 2007.

- [21] S. Peng, *Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation*, Stochastic Processes Appl, 118 (2008), 2223-2253.
- [22] S. Peng, *Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations*, Science in China Series A: Mathematics, 52 (2009), 1391-1411.
- [23] S. Peng, *Nonlinear Expectations and Stochastic Calculus under Uncertainty -with Robust Central Limit Theorem and G-Brownian Motion*, arXiv:1002.4546 v1 [math.PR] 24 Feb 2010.
- [24] S. Peng, *Stochastic Hamilton-Jacobi-Bellman equations*, SIAM J. Control Optim, 30 (1992), 284-304.
- [25] S. Peng and Z. Wu, *Fully coupled forward-BSDE and applications to optimal control*, SIAM J. Control Optim, 37 (1999), 825-843.
- [26] M. Soner, N. Touzi and J. Zhang, *Wellposedness of Second Order Backward SDEs*, arXiv:1003.6053 v1 [math.PR] 31 Mar 2010.
- [27] E. Pardoux and S. Tang, *Forward-backward stochastic differential equations and quasilinear parabolic PDEs*, Probab. Theory Relat. Fields, 114 (1999), 123-150.
- [28] J. Xu and B. Zhang, *Martingale characterization of G-Brownian motion*, Stochastic Processes and their Applications, 119 (2009), 232-248.
- [29] J. Yong and X. Zhou, *Stochastic controls: Hamiltonian systems and HJB equations*, Springer-Verlag, New York, 1999.